Objective Function

$$F = -\frac{1}{2}(1-u)\sum_{i=1}^{N} w_i \left\| (\mathbf{x}_i - z_i p_{it}\mathbf{e}_t - \sum_{k=1}^{M} p_{ik}\mathbf{e}_k) \right\|_2^2 - \frac{u}{2}\sum_{k=1}^{M} \left\| (\mathbf{e}_k - \boldsymbol{\mu}_0) \right\|_2^2 - \frac{u}{2}\left\| (\mathbf{e}_t - \boldsymbol{\mu}_0) \right\|_2^2 - \sum_{k=1}^{M}\gamma_k \sum_{i=1}^{N} p_{ik} \quad (1)$$

where, $w_i = \begin{cases} 1, \text{if } x_i \text{ is from negatively labeled bag} \\ \frac{\alpha N_n}{N_t}, \text{if } x_i \text{ is from positively labeded bag} \end{cases}$

$\gamma_k = \frac{\Gamma}{\sum_{i=1}^{N} p_{ik}^{old}}$

E-step:

Take Expectation respect to z_i:

$$\begin{aligned} E[F] &= \sum_{z_i \in \{0,1\}} \left[-\frac{1}{2}(1-u)\sum_{i=1}^{N} w_i P(z_i|\mathbf{x}_i, \boldsymbol{\theta}^{(t-1)}) \left\| \mathbf{x}_i - z_i p_{iT}\mathbf{e}_T - \sum_{k=1}^{M} p_{ik}\mathbf{e}_k \right\|_2^2 \right] \\ &\quad - \frac{u}{2}\sum_{m=1}^{M} \|\mathbf{e}_m - \boldsymbol{\mu}_0\|_2^2 - \frac{u}{2}\|\mathbf{e}_T - \boldsymbol{\mu}_0\|_2^2 - \sum_{k=1}^{M}\gamma_k \sum_{i=1}^{N} p_{ik} \end{aligned} \quad (2)$$

Where, for points from positive bags: Definition 1:

$$\begin{cases} p(z_i = 0|\mathbf{x}_i, \boldsymbol{\theta}^{(t-1)}) = e^{-\beta \left\| \mathbf{x}_i - \sum_{k=1}^{M} p_{ik}\mathbf{e}_k \right\|_2^2} \\ p(z_i = 1|\mathbf{x}_i, \boldsymbol{\theta}^{(t-1)}) = 1 - e^{-\beta \left\| \mathbf{x}_i - \sum_{k=1}^{M} p_{ik}\mathbf{e}_k \right\|_2^2} \end{cases} \quad (3)$$

Definition 2:

$$\begin{cases} p(z_i = 1|\mathbf{x}_i, \boldsymbol{\theta}^{(t-1)}) = \frac{\left| \left\| \mathbf{x}_i - \sum_{k=1}^{M} p_{ik}\mathbf{e}_k \right\|_2^2 - \left\| \mathbf{x}_i - p_{iT}\mathbf{e}_T - \sum_{k=1}^{M} p_{ik}\mathbf{e}_k \right\|_2^2 \right|}{\left\| \mathbf{x}_i - \sum_{k=1}^{M} p_{ik}\mathbf{e}_k \right\|_2^2 + \left\| \mathbf{x}_i - p_{iT}\mathbf{e}_T - \sum_{k=1}^{M} p_{ik}\mathbf{e}_k \right\|_2^2} \\ p(z_i = 0|\mathbf{x}_i, \boldsymbol{\theta}^{(t-1)}) = 1 - \frac{\left| \left\| \mathbf{x}_i - \sum_{k=1}^{M} p_{ik}\mathbf{e}_k \right\|_2^2 - \left\| \mathbf{x}_i - p_{iT}\mathbf{e}_T - \sum_{k=1}^{M} p_{ik}\mathbf{e}_k \right\|_2^2 \right|}{\left\| \mathbf{x}_i - \sum_{k=1}^{M} p_{ik}\mathbf{e}_k \right\|_2^2 + \left\| \mathbf{x}_i - p_{iT}\mathbf{e}_T - \sum_{k=1}^{M} p_{ik}\mathbf{e}_k \right\|_2^2} \end{cases} \quad (4)$$

where β is a scaling parameter and $\left\| \mathbf{x}_i - \sum_{k=1}^{M} p_{ik}\mathbf{e}_k \right\|_2^2$ is the approximation redisual between $\mathbf{x}_i$ and its representation using only background endmembers; $\left\| \mathbf{x}_i - p_{iT}\mathbf{e}_T - \sum_{k=1}^{M} p_{ik}\mathbf{e}_k \right\|_2^2$ is the approximation redisual between $\mathbf{x}_i$ and its representation both using target endmember and background endmembers.

M-step:
P update equation:

For points from positive bags:

$$F^+ = \sum_{i=1}^{N^+} \Bigg[- P(z_i=0)\frac{1}{2}(1-u)w_i \Big\| (\mathbf{x}_i - \sum_{k=1}^{M} p_{ik}\mathbf{e}_k) \Big\|_2^2 - P(z_i=1)\frac{1}{2}(1-u)w_i \Big\| (\mathbf{x}_i - P_{it}\mathbf{e}_t - \sum_{k=1}^{M} p_{ik}\mathbf{e}_k \Big\|_2^2 \Bigg]$$
$$- \sum_i \lambda_i^+ (p_{it} + \sum_{k=1}^{M} p_{ik} - 1) - \sum_{k=1}^{M} \gamma_k \sum_{i=1}^{N^+} p_{ik} \tag{5}$$

Define:

$$\mathbf{P}_i = \begin{bmatrix} p_{it} \\ p_{i1} \\ p_{i2} \\ \vdots \\ p_{iM} \end{bmatrix} = \begin{bmatrix} p_{it} \\ \mathbf{P}_i^- \end{bmatrix}, \mathbf{E} = \begin{bmatrix} \mathbf{e}_t & \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_M \end{bmatrix}, E^- = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_M \end{bmatrix}$$

Where $\boldsymbol{E}$ is the endmember matrix whose column corresponds to an endmember spectrum. $\boldsymbol{E}^-$ is a subset of $\boldsymbol{E}$ which accounts for constituent background endmembers. Similarly $\boldsymbol{P}_i$ is proportion vector for point $\boldsymbol{x}_i$ and $\boldsymbol{P}_i^-$ is a subset of $\boldsymbol{P}_i$, which accounts for the proportion values with respect to background endnembers. For points from negative bags, p_{it} is constrained to 0, so $\boldsymbol{P}_i = \begin{bmatrix} 0 \\ \boldsymbol{P}_i^- \end{bmatrix}$.

Take partial derivitive on Equation **??** with respect to p_{it} and p_{ik}, respectively:

$$\frac{\partial F^+}{\partial p_{it}} = P(z_i=1)(1-u)w_i(-1)e_t^T(\mathbf{x}_i - p_{it}\mathbf{e}_t - \sum_{k=1}^{M} p_{ik}\mathbf{e}_k) + \lambda_i^+$$

$$\frac{\partial F^+}{\partial p_{ik}} = P(z_i=0)(1-u)w_i(-1)\mathbf{e}_k^T(\mathbf{x}_i - \sum_{k=1}^{M} p_{ik}\mathbf{e}_k)$$
$$+ P(z_i=1)(1-u)w_i(-1)\mathbf{e}_k^T(x_i - p_{it}\mathbf{e}_t - \sum_{k=1}^{M} p_{ik}e_k) + \lambda_i^+ + \gamma_k$$

Denote: $a = (1-u)w_i(-1)$, and rewrite the above two expressions in a consistent matrix from:

$$\frac{\partial F^+}{\partial p_{it}} = aP(z_i=0)\mathbf{0}_{d\times 1}^T\big(\mathbf{x}_i - \begin{bmatrix} 0 & \mathbf{E}^- \end{bmatrix} P_i\big) + aP(z_i=1)\mathbf{e}_t^T(\mathbf{x}_i - \mathbf{E}P_i) + \lambda_i^+ + 0$$

$$\frac{\partial F^+}{\partial p_{ik}} = aP(z_i=0)\mathbf{e}_k^T\big(\mathbf{x}_i - \begin{bmatrix} 0 & \mathbf{E}^- \end{bmatrix} P_i\big) + aP(z_i=1)\mathbf{e}_k^T(\mathbf{x}_i - \mathbf{E}P_i) + \lambda_i^+ + \gamma_k$$

⇒combine into vector form:

$$\begin{aligned}\frac{\partial F^+}{\partial \mathbf{P}_i} =& aP(z_i=0)\begin{bmatrix}0 & \mathbf{E}^-\end{bmatrix}^T\left(\mathbf{x}_i - \begin{bmatrix}0 & \mathbf{E}^-\end{bmatrix}P_i\right) + aP(z_i=1)\mathbf{E}^T(\mathbf{x}_i - \mathbf{E}P_i) + \lambda_i^+\mathbf{1}_{(M+1)\times 1} + \begin{bmatrix}0\\ \mathbf{V}\end{bmatrix}\\ =&\left[aP(z_i=0)\begin{bmatrix}0 & \mathbf{E}^-\end{bmatrix}^T + aP(z_i=1)\mathbf{E}^T\right]\mathbf{x}_i\\ &-\left\{aP(z_i=0)\begin{bmatrix}0 & \mathbf{E}^-\end{bmatrix}^T\begin{bmatrix}0 & \mathbf{E}^-\end{bmatrix} + aP(z_i=1)\mathbf{E}^T\mathbf{E}\right\}P_i + \lambda_i^+\mathbf{1}_{(M+1)\times 1} + \begin{bmatrix}0\\ \mathbf{V}\end{bmatrix} = 0\end{aligned}$$

Where: $\mathbf{V} = \begin{bmatrix}\gamma_1\\ \gamma_2\\ \vdots\\ \gamma_k\end{bmatrix}$

⇒Solve for $\boldsymbol{P}_i$:

$$\begin{aligned}\mathbf{P}_i =&\left\{P(z_i=0)\begin{bmatrix}0 & \mathbf{E}^-\end{bmatrix}^T\begin{bmatrix}0 & \mathbf{E}^-\end{bmatrix} + P(z_i=1)\mathbf{E}^T\mathbf{E}\right\}^{-1}\cdot\\ &\left\{\left[P(z_i=0)\begin{bmatrix}\mathbf{0}_{d\times 1} & \mathbf{E}^-\end{bmatrix}^T + P(z_i=1)\mathbf{E}^T\right]x_i + \mathbf{1}_{(M+1)\times 1}\frac{\lambda_i^+}{a} + \frac{1}{a}\begin{bmatrix}0\\ \mathbf{V}\end{bmatrix}\right\}\end{aligned} \tag{6}$$

use the sum to one constraint: $\mathbf{1}_{1\times(M+1)}P_i = 1$

⇒

$$\lambda_i^+ = \frac{a\left(1 - \mathbf{1}_{1\times(M+1)}\left\{P(z_i=0)\begin{bmatrix}\mathbf{0}_{d\times 1} & \mathbf{E}^-\end{bmatrix}^T\begin{bmatrix}\mathbf{0}_{d\times 1} & \mathbf{E}^-\end{bmatrix} + P(z_i=1)\mathbf{E}^T\mathbf{E}\right\}^{-1}\cdot\left\{\left[P(z_i=0)\begin{bmatrix}\mathbf{0}_{d\times 1} & \mathbf{E}^-\end{bmatrix}^T + P(z_i=1)\mathbf{E}^T\right]\mathbf{x}_i + \frac{1}{a}\begin{bmatrix}0\\ \mathbf{V}\end{bmatrix}\right\}\right)}{\mathbf{1}_{1\times(M+1)}\left\{P(z_i=0)\begin{bmatrix}\mathbf{0}_{d\times 1} & \mathbf{E}^-\end{bmatrix}^T\begin{bmatrix}\mathbf{0}_{d\times 1} & \mathbf{E}^-\end{bmatrix} + P(z_i=1)\mathbf{E}^T\mathbf{E}\right\}^{-1}\mathbf{1}_{(M+1)\times 1}} \tag{7}$$

Equation **??** → Equation **??**

$$\begin{aligned}\mathbf{P}_i =& \left\{P(z_i=0)\begin{bmatrix}\mathbf{0}_{d\times 1} & \mathbf{E}^-\end{bmatrix}^T\begin{bmatrix}\mathbf{0}_{d\times 1} & \mathbf{E}^-\end{bmatrix} + P(z_i=1)\mathbf{E}^T\mathbf{E}\right\}^{-1}\cdot\\ &\left\{\left[P(z_i=0)\begin{bmatrix}\mathbf{0}_{d\times 1} & \mathbf{E}^-\end{bmatrix}^T + P(z_i=1)\mathbf{E}^T\right]\mathbf{x}_i + \frac{1}{a}\begin{bmatrix}0\\ \mathbf{V}\end{bmatrix} + 1_{(M+1)\times 1}\cdot\right.\\ &\left.\frac{1 - \mathbf{1}_{1\times(M+1)}\left\{P(z_i=0)\begin{bmatrix}\mathbf{0}_{d\times 1} & \mathbf{E}^-\end{bmatrix}^T\begin{bmatrix}\mathbf{0}_{d\times 1} & \mathbf{E}^-\end{bmatrix} + P(z_i=1)\mathbf{E}^T\mathbf{E}\right\}^{-1}\cdot\left\{\left[P(z_i=0)\begin{bmatrix}\mathbf{0}_{d\times 1} & \mathbf{E}^-\end{bmatrix}^T + P(z_i=1)\mathbf{E}^T\right]x_i + \frac{1}{a}\begin{bmatrix}0\\ \mathbf{V}\end{bmatrix}\right\}}{\mathbf{1}_{1\times(M+1)}\left\{P(z_i=0)\begin{bmatrix}\mathbf{0}_{d\times 1} & \mathbf{E}^-\end{bmatrix}^T\begin{bmatrix}\mathbf{0}_{d\times 1} & \mathbf{E}^-\end{bmatrix} + P(z_i=1)\mathbf{E}^T\mathbf{E}\right\}^{-1}\mathbf{1}_{(M+1)\times 1}}\right\}\end{aligned} \tag{8}$$

Here, it is difficult to write $\mathbf{P}_i$ in matrix form because in $\left\{P(z_i=0)\begin{bmatrix}\mathbf{0}_{d\times 1} & \mathbf{E}^-\end{bmatrix}^T\begin{bmatrix}\mathbf{0}_{d\times 1} & \mathbf{E}^-\end{bmatrix} + P(z_i = 1)\mathbf{E}^T\mathbf{E}\right\}^{-1}$, $\mathbf{P}_i$ is related to $P(z_i)$. The proportion of points from positive bags must be updated point by point.

For points from negative bags:

$$F^{-}=-\sum_{i=1}^{N^{-}}\left[\frac{(1-u)}{2}\left\|(\mathbf{x}_i-\sum_{k=1}^{M}p_{ik}\mathbf{e}_k)\right\|_2^2\right]-\sum_{i}\lambda_i^{-}(\sum_{k=1}^{M}p_{ik}-1)-\sum_{k=1}^{M}\gamma_k\sum_{i=1}^{N-}p_{ik} \tag{9}$$

Take partial derivitive on Equation **??** with respect to p_{ik}:

$$\frac{\partial F^{-}}{\partial p_{ik}}=(1-u)(-1)\mathbf{e}_k^T(x_i-\sum_{k=1}^{M}p_{ik}\mathbf{e}_k)+\lambda_i^{-}+\gamma_k$$

Vector from:

$$\frac{\partial F^{-}}{\partial \mathbf{P}_i^{-}}=(1-u)(-1)\mathbf{E}^{-T}(\mathbf{x}_i-\mathbf{E}^{-}\mathbf{P}_i^{-})+\lambda_i^{-}\mathbf{1}_{M\times 1}+\mathbf{V}=0 \tag{10}$$

Solve for $\boldsymbol{P}_i^{-}$: $\Rightarrow$

$$\mathbf{P}_i^{-}=(\mathbf{E}^{-T}\mathbf{E}^{-})^{-1}\left(\mathbf{E}^{-T}\mathbf{x}_i-\frac{1}{1-u}\mathbf{V}\right)-\frac{\lambda_i^{-}}{1-u}(\mathbf{E}^{-T}\mathbf{E}^{-})^{-1}\cdot\mathbf{1}_{M\times 1} \tag{11}$$

use the sum to one constraint:

$$\mathbf{1}_{1\times M}\mathbf{P}_i^{-}=1$$

$\Rightarrow$

$$\lambda_i^{-}=-\frac{(1-u)\left(1-\mathbf{1}_{1\times M}(\mathbf{E}^{-T}\mathbf{E}^{-})^{-1}\left(\mathbf{E}^{-T}\mathbf{x}_i+\frac{1}{a}\mathbf{V}\right)\right)}{\mathbf{1}_{1\times M}(\mathbf{E}^{-T}\mathbf{E}^{-})^{-1}\mathbf{1}_{M\times 1}} \tag{12}$$

Equation **??** $\rightarrow$ Equation **??**

$$\Rightarrow \mathbf{P}_i^{-}=(\mathbf{E}^{-T}\mathbf{E}^{-})^{-1}\left[\mathbf{E}^{-T}\mathbf{x}_i+\frac{1}{a}\mathbf{V}+1_{M\times 1}\frac{1-\mathbf{1}_{1\times M}(\mathbf{E}^{-T}\mathbf{E}^{-})^{-1}\left(\mathbf{E}^{-T}\mathbf{x}_i+\frac{1}{a}\mathbf{V}\right)}{\mathbf{1}_{1\times M}(\mathbf{E}^{-T}\mathbf{E}^{-})^{-1}\mathbf{1}_{M\times 1}}\right]$$

Matrix form:

$$\mathbf{P}^{-}=(\mathbf{E}^{-T}\mathbf{E}^{-})^{-1}\left[\mathbf{E}^{-T}\mathbf{X}^{-}+\frac{1}{a}\text{repmat}(\mathbf{V},1,N^{-})+\mathbf{1}_{M\times 1}\cdot\frac{1-\mathbf{1}_{1\times M}(\mathbf{E}^{-T}\mathbf{E}^{-})^{-1}\left(\mathbf{E}^{-T}\mathbf{X}^{-}+\frac{1}{a}\text{repmat}(\mathbf{V},1,N^{-})\right)}{\mathbf{1}_{1\times M}(\mathbf{E}^{-T}\mathbf{E}^{-})^{-1}\mathbf{1}_{M\times 1}}\right] \tag{13}$$

$\Rightarrow$In each iteration:

$$\mathbf{P}=\begin{bmatrix}\mathbf{P}^{+} & \mathbf{0}_{1\times N^-} \\ & \mathbf{P}^{-}\end{bmatrix}_{(M+1)\times N} \tag{14}$$

E update:

$$F = \sum_{i=1}^{N}\left[P(z_i=0)\frac{1}{2}(1-u)w_i\left\|(\mathbf{x}_i-\sum_{k=1}^{M}p_{ik}\mathbf{e}_k)\right\|_2^2+P(z_i=1)\frac{1}{2}(1-u)w_i\left\|(\mathbf{x}_i-P_{it}\mathbf{e}_t-\sum_{k=1}^{M}p_{ik}\mathbf{e}_k\right\|_2^2\right] + \frac{u}{2}\sum_{k=1}^{M}\left\|(\mathbf{e}_k-\boldsymbol{\mu}_0)\right\|_2^2+\frac{u}{2}\left\|(\mathbf{e}_t-\boldsymbol{\mu}_0)\right\|_2^2 \quad (15)$$

Add norm to 1 constraint:

$$F = \sum_{i=1}^{N^+}\left[P(z_i=0)\frac{1}{2}(1-u)w_i\left\|(\mathbf{x}_i-\sum_{k=1}^{M}p_{ik}\mathbf{e}_k)\right\|_2^2+P(z_i=1)\frac{1}{2}(1-u)w_i\left\|(\mathbf{x}_i-P_{it}\mathbf{e}_t-\sum_{k=1}^{M}p_{ik}\mathbf{e}_k\right\|_2^2\right] \quad (16)$$
$$+\frac{1}{2}(1-u)\sum_{i=1}^{N^-}\left[\left\|(\mathbf{x}_i-\sum_{k=1}^{M}p_{ik}\mathbf{e}_k)\right\|_2^2\right]+\frac{u}{2}\sum_{k=1}^{M}\left\|(\mathbf{e}_k-\boldsymbol{\mu}_0)\right\|_2^2+\frac{u}{2}\left\|(\mathbf{e}_t-\boldsymbol{\mu}_0)\right\|_2^2+\sum_{k=1}^{M}\lambda_k(\|\mathbf{e}_k\|_2^2-1^2)+\lambda_t(\|\mathbf{e}_t\|_2^2-1^2)$$

Take partial derivitive on Equation **??** with respect to $\boldsymbol{e}_t$ and $\boldsymbol{e}_k$:

$$\frac{\partial F}{\partial \mathbf{e}_t} = \sum_{i=1}^{N^+}\left[P(z_i=1)(-1)(1-u)w_ip_{it}(\mathbf{x}_i-p_{it}\mathbf{e}_t-\sum_{k=1}^{M}p_{ik}\mathbf{e}_k)\right]+u(\mathbf{e}_t-\boldsymbol{\mu}_0)+2\lambda_t\cdot\mathbf{e}_t \quad (17)$$

$$\frac{\partial F}{\partial \mathbf{e}_k} = \sum_{i=1}^{N^+}\left[P(z_i=0)(-1)(1-u)w_ip_{ik}(\mathbf{x}_i-\sum_{k=1}^{M}p_{ik}\mathbf{e}_k)+P(z_i=1)(-1)(1-u)w_ip_{ik}(\mathbf{x}_i-p_{it}\mathbf{e}_t-\sum_{k=1}^{M}p_{ik}\mathbf{e}_k)\right] +(1-u)\sum_{i=1}^{N^-}\left[-p_{ik}(\mathbf{x}_i-\sum_{k=1}^{M}p_{ik}\mathbf{e}_k)\right]+u(\mathbf{e}_k-\boldsymbol{\mu}_0)+2\lambda_k\cdot\mathbf{e}_k \quad (18)$$

Solve for $\mathbf{e}_t$ from Equation **??**:

$$\text{Equation } ?? = 0 \Rightarrow \sum_{i=1}^{N^+}\left[P(z_i=1)(-1)(1-u)w_ip_{it}(\mathbf{x}_i-p_{it}\mathbf{e}_t-\sum_{k=1}^{M}p_{ik}\mathbf{e}_k)\right]+u(\mathbf{e}_t-\boldsymbol{\mu}_0)+2\lambda_t\cdot\mathbf{e}_t=0 \quad (19)$$

$$\Rightarrow \sum_{i=1}^{N^+}\left[P(z_i=1)(1-u)w_ip_{it}(\mathbf{x}_i-\sum_{k=1}^{M}p_{ik}\mathbf{e}_k)\right]+u\boldsymbol{\mu}_0=\Big\{\sum_{i=1}^{N+}[P(z_i=1)(1-u)w_ip_{it}^2]+u+2\lambda_t\Big\}\mathbf{e}_t \quad (20)$$

$$\Rightarrow \mathbf{e}_t = \frac{\sum_{i=1}^{N^+}\left[P(z_i=1)(1-u)w_ip_{it}(\mathbf{x}_i-\sum_{k=1}^{M}p_{ik}\mathbf{e}_k)\right]+u\boldsymbol{\mu}_0}{\sum_{i=1}^{N+}[P(z_i=1)(1-u)w_ip_{it}^2]+u+2\lambda_t} \quad (21)$$

use norm to 1 constraint:

$$1_{1\times d}\cdot\boldsymbol{e}_t.\hat{\ }2=1^2$$

$$\lambda_t = \frac{1}{2}\left(\left\| \sum_{i=1}^{N^+} \left[P(z_i=1)(1-u)w_i p_{it}(\mathbf{x}_i - \sum_{k=1}^{M} p_{ik}\mathbf{e}_k)\right] + u\boldsymbol{\mu}_0 \right\| - \sum_{i=1}^{N+}[P(z_i=1)(1-u)w_i p_{it}^2] - u \right) \tag{22}$$

Equation **??** → Equation **??**

$$\Rightarrow \mathbf{e}_t = \frac{\sum_{i=1}^{N^+}\left[P(z_i=1)(1-u)w_i p_{it}(\mathbf{x}_i - \sum_{k=1}^{M} p_{ik}\mathbf{e}_k)\right] + u\boldsymbol{\mu}_0}{\left\| \sum_{i=1}^{N^+}\left[P(z_i=1)(1-u)w_i p_{it}(\mathbf{x}_i - \sum_{k=1}^{M} p_{ik}\mathbf{e}_k)\right] + u\boldsymbol{\mu}_0 \right\|} \tag{23}$$

Solve for $\mathbf{e}_k$ from Equation **??**:

$$\text{Equation } \textbf{??} = 0 \Rightarrow \quad \sum_{i=1}^{N^+}\left[P(z_i=0)(-1)(1-u)w_i p_{ik}(\mathbf{x}_i - \sum_{k=1}^{M} p_{ik}\mathbf{e}_k) + P(z_i=1)(-1)(1-u)w_i p_{ik}(\mathbf{x}_i - p_{it}\mathbf{e}_t - \sum_{k=1}^{M} p_{ik}\mathbf{e}_k)\right]$$
$$+(1-u)\sum_{i=1}^{N^-}\left[-p_{ik}(\mathbf{x}_i - \sum_{k=1}^{M} p_{ik}\mathbf{e}_k)\right] + u(\mathbf{e}_k - \boldsymbol{\mu}_0) + 2\lambda_k \cdot \mathbf{e}_k = 0 \tag{24}$$

$$\Rightarrow \quad \sum_{i=1}^{N^+}\left[P(z_i=0)(1-u)w_i p_{ik}^2\mathbf{e}_k + P(z_i=1)(1-u)w_i p_{ik}^2\mathbf{e}_k\right] + \sum_{i=1}^{N^-}\left[(1-u)p_{ik}^2\mathbf{e}_k\right] + u\mathbf{e}_k + 2\lambda_k\mathbf{e}_k =$$
$$\sum_{i=1}^{N^+}\left[P(z_i=0)(1-u)w_i p_{ik}(\mathbf{x}_i - \sum_{l=1,l\neq k}^{M} p_{il}\mathbf{e}_l) + P(z_i=1)(1-u)w_i p_{ik}(\mathbf{x}_i - p_{it}\mathbf{e}_t - \sum_{l=1,l\neq k}^{M} p_{il}\mathbf{e}_l)\right]$$
$$+\sum_{i=1}^{N^-}\left[(1-u)p_{ik}(\mathbf{x}_i - \sum_{l=1,l\neq k}^{M} p_{il}\mathbf{e}_l)\right] + u\boldsymbol{\mu}_0 \tag{25}$$

$$\Rightarrow \quad \sum_{i=1}^{N^+}\left[(1-u)w_i p_{ik}^2\mathbf{e}_k\right] + \sum_{i=1}^{N^-}\left[(1-u)p_{ik}^2\mathbf{e}_k\right] + u\mathbf{e}_k + 2\lambda_k\mathbf{e}_k =$$
$$\sum_{i=1}^{N^+}\left[P(z_i=0)(1-u)w_i p_{ik}(\mathbf{x}_i - \sum_{l=1,l\neq k}^{M} p_{il}\mathbf{e}_l) + P(z_i=1)(1-u)w_i p_{ik}(\mathbf{x}_i - p_{it}\mathbf{e}_t - \sum_{l=1,l\neq k}^{M} p_{il}\mathbf{e}_l)\right]$$
$$+\sum_{i=1}^{N^-}\left[(1-u)p_{ik}(\mathbf{x}_i - \sum_{l=1,l\neq k}^{M} p_{il}\mathbf{e}_l)\right] + u\boldsymbol{\mu}_0 \tag{26}$$

$$
\begin{aligned}
\Rightarrow \mathbf{e}_k &= \left\{\sum_{i=1}^{N^+}\left[P(z_i=0)(1-u)w_i p_{ik}(\mathbf{x}_i - \sum_{l=1,l\neq k}^{M} p_{il}\mathbf{e}_l) + P(z_i=1)(1-u)w_i p_{ik}(\mathbf{x}_i - p_{it}\mathbf{e}_t - \sum_{l=1,l\neq k}^{M} p_{il}\mathbf{e}_l)\right]\right. \\
&\left. + \sum_{i=1}^{N^-}\left[(1-u)p_{ik}(\mathbf{x}_i - \sum_{l=1,l\neq k}^{M} p_{il}\mathbf{e}_l)\right] + u\boldsymbol{\mu}_0\right\} \cdot \left\{\sum_{i=1}^{N^+}\left[(1-u)w_i p_{ik}^2\right] + \sum_{i=1}^{N^-}\left[(1-u)p_{ik}^2\right] + u + 2\lambda_k\right\}^{-1} \quad (27)
\end{aligned}
$$

use norm to 1 constraint:

$$
1_{1\times d} \cdot \boldsymbol{e}_k.\hat{}2 = 1^2
$$

$$
\begin{aligned}
\lambda_k &= \frac{1}{2}\left\{\left\|\sum_{i=1}^{N^+}\left[P(z_i=0)(1-u)w_i p_{ik}(\mathbf{x}_i - \sum_{l=1,l\neq k}^{M} p_{il}\mathbf{e}_l) + P(z_i=1)(1-u)w_i p_{ik}(\mathbf{x}_i - p_{it}\mathbf{e}_t - \sum_{l=1,l\neq k}^{M} p_{il}\mathbf{e}_l)\right]\right.\right. \\
&\left.\left. + \sum_{i=1}^{N^-}\left[(1-u)p_{ik}(\mathbf{x}_i - \sum_{l=1,l\neq k}^{M} p_{il}\mathbf{e}_l)\right] + u\mu_0\right\| - \sum_{i=1}^{N^+}\left[(1-u)w_i p_{ik}^2\right] - \sum_{i=1}^{N^-}\left[(1-u)p_{ik}^2\right] - u - 2\lambda_k\right\} \quad (28)
\end{aligned}
$$

Equation **??** → Equation **??**

$$
\begin{aligned}
\Rightarrow \mathbf{e}_k &= \left\{\sum_{i=1}^{N^+}\left[P(z_i=0)(1-u)w_i p_{ik}(\mathbf{x}_i - \sum_{l=1,l\neq k}^{M} p_{il}\mathbf{e}_l) + P(z_i=1)(1-u)w_i p_{ik}(\mathbf{x}_i - p_{it}\mathbf{e}_t - \sum_{l=1,l\neq k}^{M} p_{il}\mathbf{e}_l)\right]\right. \\
&\left. + \sum_{i=1}^{N^-}\left[(1-u)p_{ik}(\mathbf{x}_i - \sum_{l=1,l\neq k}^{M} p_{il}\mathbf{e}_l)\right] + u\boldsymbol{\mu}_0\right\} \cdot \\
&\left\|\sum_{i=1}^{N^+}\left[P(z_i=0)(1-u)w_i p_{ik}(\mathbf{x}_i - \sum_{l=1,l\neq k}^{M} p_{il}\mathbf{e}_l) + P(z_i=1)(1-u)w_i p_{ik}(\mathbf{x}_i - p_{it}\mathbf{e}_t - \sum_{l=1,l\neq k}^{M} p_{il}\mathbf{e}_l)\right]\right. \\
&\left. + \sum_{i=1}^{N^-}\left[(1-u)p_{ik}(\mathbf{x}_i - \sum_{l=1,l\neq k}^{M} p_{il}\mathbf{e}_l)\right] + u\boldsymbol{\mu}_0\right\|^{-1} \quad (29)
\end{aligned}
$$