Lecture 4: Charged Particle Motion

Charged particles move in response to electric and magnetic fields by the Lorentz Force

\[ \vec{F} = m \vec{a} = m \frac{d\vec{v}}{dt} \]

- \( \vec{F}_e = q \vec{E} \)
- \( \vec{F}_m = q (\nabla \times \vec{B}) \)

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Case: \( B = 0, E \) in x-direction

\[ m \frac{d\vec{v}}{dt} = q \vec{E} = q E_x \vec{x} \]

so \( a_y = a_z = 0 \)

\[ m \dot{v}_x = q E_x \]

\[ v_x(t) = \frac{q}{m} E_x t + v_x(0) \]

Case: \( E = 0, B \) in z-direction

\[ m \frac{d\vec{v}}{dt} = q (\nabla \times \vec{B}) \rightarrow \vec{v}_x \vec{B} = \begin{vmatrix} x & y & z \\ y_x & y_y & y_z \\ 0 & 0 & B_z \end{vmatrix} \]

\[ = \dot{x} v_y B_z - \dot{y} v_x B_z \]
So

\[ m \ddot{v}_x = q v_y B_z \]
\[ m \ddot{v}_y = -q v_x B_z \]
\[ m \ddot{v}_z = 0 \quad \rightarrow \quad v_z = \text{constant} \]

define \( \omega_c = \frac{q B}{m} \) cyclotron frequency

\[ \dot{v}_x = \pm \omega_c v_y \]
\[ \dot{v}_y = \mp \omega_c v_x \]

take time derivative of one equation and substitute the other

\[ \ddot{v}_y = \mp \omega_c \dot{v}_x = \mp \omega_c (\pm \omega_c v_y) \]
\[ = -\omega_c^2 v_y \]
\[ \dot{v}_x = -\omega_c^2 v_x \]

defines a harmonic oscillator at the cyclotron frequency

\[ v_x = \pm v_{\perp} \sin \omega_c t \]
\[ v_y \propto \dot{v}_x \rightarrow \quad v_y = v_{\perp} \cos \omega_c t \]

depending on the sign of \( q \)
Can show that the motion is a circle by integrating the velocities to find the x and y positions

\[(x-x_0)^2 + (y-y_0)^2 = \left(\frac{V_\perp}{\omega c}\right)^2\]

define the Larmor Radius

\[r_L = \frac{V_\perp}{\omega c}\]

Motion for + and - charge

Note: Moving charge makes current, current so as to cancel applied field

Case: B in z-direction, E in x-direction

\[m\dot{V} = q \left(E + \vec{V} \times \vec{B}\right)\]
\[= \pm q \left(E + \vec{v} \times \vec{B}\right)\]
\[m\dot{V}_x = \pm q \left(E_x + v_y B_z\right)\]
\[m\dot{V}_y = \pm q v_x B_z\]
\[m\dot{V}_z = 0\]
\[ \dot{v}_x = \pm \frac{|q|}{m} E_x \pm \omega_c v_y \]

\[ \dot{v}_y = \mp \omega_c v_x \]

take a time derivative of the $y$-acceleration

\[ \ddot{v}_y = \mp \omega_c \dot{v}_x = \mp \omega_c (\pm \frac{|q|}{m} E_x \pm \omega_c v_y) \]

\[ = -\omega_c^2 \left( \frac{E_x}{B_z} + v_y \right) \]

\[ \text{take } \ddot{y}_y = \frac{E_x}{B_z} + v_y \]

then \[ \ddot{v}_y = -\omega_c^2 \ddot{y}_y \]

\[ \dot{v}_x = -\omega_c^2 v_x \]

Solution

\[ v_x = v_\perp e^{i\omega_c t} \]

\[ v_y = \pm i v_\perp e^{i\omega_c t} - \frac{E_x}{B_z} \]

motion in $x$-$y$ plane, positive and negative charges move in same direction, direction is $E \times B$

constant velocity drift on top of cyclotron motion, perpendicular to $z$ and $x$
for any E and B direction

\[ \vec{v}_{\text{ExB}} = \frac{\vec{E} \times \vec{B}}{|B|^2} \]

**Notes**

+ It is a constant velocity drift - there is a whole class of these
+ independent of perpendicular velocity, charge, or mass - same speed for massive ions and light electrons
+ Important in magnetrons (microwave oven RF source) and MITL-magnetically insulated transmission lines

**Particle current**

+ So, charges can move under the influence of fields
+ Moving charge means current
+ say we have a distribution of charges, \( n \) [#/vol], with charge, \( q \), moving at velocity, \( v \)

\[ \vec{j} = q \vec{n} \vec{v} = \rho \vec{v} \]

\[ J = \int \vec{j} \cdot d\vec{A} \]

+ so, with current density and velocity, we can determine the charge density of an electron beam.

**Relativistic motion**

Let's back up, for non-relativistic particles, if a force acts on a particle, its velocity can change

\[ \vec{F} = m \vec{v} \]

defining momentum, \( p \), we could also write

\[ \vec{F} = \frac{d}{dt} \vec{p} \quad ; \quad \vec{p} = m \vec{v} \]
particle kinetic energy is

\[ T = \frac{1}{2} m v^2 \]

if we accelerated the charge through some potential, \( V \)

\[ T_{\text{final}} = q V = \frac{1}{2} m v^2 \]

This works fine if the particle velocity is not near the speed of light. As we approach the speed of light, some corrections must be made.

define two parameters

\[ \beta = \frac{v}{c} \quad \text{and} \quad \gamma = \frac{1}{\sqrt{1 - \beta^2 c^2}} \]

so

\[ \gamma = \frac{1}{\sqrt{1 - \beta^2}} \quad \text{and} \quad \beta = (1 - \frac{1}{\gamma^2})^{1/2} \]

the force law in terms of momentum still works, with one addition

\[ \vec{F} = \dot{\vec{p}} \quad \text{with} \quad \vec{p} = \gamma m_o \vec{v} \]

\[ = m_o \vec{v} \gamma + \gamma m_o \vec{\gamma} \]

\( m_o \) is a constant, the rest mass, and can think of effect as the particle mass increasing near the speed of light

\[ m = \gamma m_o \]

To keep changes in energy equal to force acting through a distance, the particle energy must be

\[ \varepsilon = \gamma m_o c^2 \]
When a particle is at rest, \( \gamma = 1 \), so the particle must have a rest energy

\[
\mathcal{E}_0 = m_o c^2
\]

Kinetic energy (which we add via electric or electromagnetic forces) is the difference between the energy and the rest energy

\[
T = \mathcal{E} - \mathcal{E}_0 = \gamma m_o c^2 - m_o c^2 = (\gamma - 1)m_o c^2
\]

A couple of other useful formulas

\[
\mathcal{E} = \sqrt{\gamma^2 m_o^2 c^4 + m_o^2 c^4} = \sqrt{m_o^2 c^4 (1 + \frac{v^2}{c^2} + \ldots)}\]

\[
\vec{V} = \frac{\vec{p}}{\mathcal{E}} = \frac{\vec{p}}{\sqrt{\gamma^2 m_o^2 c^4 + m_o^2 c^4}}
\]

so when do we need to use the relativistic equations for motion and energy? Well look at the total particle energy \( (T+m_o c^2) \), and expand for \( v/c \ll 1 \)

\[
\mathcal{E} = \frac{m_o c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \approx m_o c^2 \left(1 + \frac{v^2}{2c^2} + \ldots\right)
\]

Then \( T \) becomes

\[
T = \mathcal{E} - \mathcal{E}_0 \approx (m_o c^2 + \frac{m_o v^2}{2} + \ldots) - m_o c^2
\]

\[
= \frac{m_o v^2}{2}
\]

So, non-relativistic when \( v/c \ll 1 \).
Equivalently, when \( eV \ll m_o c^2 \).