Lecture 10: Charged Particle Optics

We know charged particles are affected by electric and magnetic fields, through the Lorentz force law. By configuring electric and magnetic fields, we can produce lenses.

Purposes of charged particle lenses
1) To image, as in ordinary optics
2) To focus, to deflection the particles to a single point
3) To confine, an array of lenses can confine a beam to a small space

Paraxial Approximation of E and B
+ Assume a cylindrical beam
+ Assume E and B are azimuthally symmetric
+ Assume $E_r \ll E_z$, and $B_r \ll B_z$

Can show that the radial fields depend linearly on radius

$$E_r (r,z) \approx - \left( \frac{r}{z} \right) \frac{\partial E_z (0,z)}{\partial z}$$

$$B_r (r,z) \approx - \left( \frac{r}{z} \right) \frac{\partial B_z (0,z)}{\partial z}$$

These linearly varying radial fields will produce focusing, particularly if we apply the paraxial approximation to the particle paths, so that $v_r \ll v_z$. We also assume we can convert time derivatives into space derivatives --

$$\frac{1}{\tau} \Rightarrow \frac{v_z}{\partial z} \frac{d}{dz}$$
Basics of charged particle optics

The Thick Lens

f_1 and f_2 -- focal lengths
H_1 and H_2 -- principle planes
F_1 and F_2 -- locate principle planes relative to lens midplane
p and q -- object and image distances from principle planes
P and Q -- object and image distances from lens mid-plane

M=x_2/x_1 -- linear magnification
m=\theta_2/\theta_1 -- angular magnification (\theta was the divergence of particles emitted from a point, in the beam brightness calc, etc.)

We can show for thick lenses

\[ M = \frac{f_z - q}{f_z} = \frac{f_i}{f_i - P} \]
\[ p = F_1 - \frac{f_1}{M} \]
\[ Q = F_2 - M f_z \]
\[ (P-f_i)(q-f_z) = (P-F_1)(Q-F_z) = f_i f_z \]
\[ \frac{f_i}{P} + \frac{f_z}{Q} = 1 \]
Electrostatic aperture lens

\[ E_a = \frac{v_z - v_1}{a} \]

\[ E_b = \frac{v_3 - v_2}{b} \]

\[ f = \frac{4 \cdot V_2}{E_b - E_a} \]

\[ \theta_2 = \theta_1 \frac{f - a}{f} \]

Note that this is a thin lens, the principle planes coincide at the lense mid plane, and \( f_1 = f_2 = F_1 = F_2 \).

The radial acceleration can be found from the following

\[ \frac{dv_r}{dt} = v_z \frac{dv_r}{dz} = \frac{q E_r}{m_0} \]

\[ \frac{dv_r}{dz} \sim - \left( \frac{q}{2m_0 v_z} \right) r \frac{dE_z(0, z)}{dz} \]

the ratio of exit radial velocity to exit axial velocity is \( v_{za} \rightarrow v \) at aperture

\[ \frac{v_{r_{zf}}}{v_{z_{zf}}} \sim \frac{r}{f} \sim - \left( \frac{q}{2m_0} \right) \frac{E_b - E_a}{V_{za} V_{zf}} r \]
Electrostatic immersion lens

Beam travels from a metal cylinder at one potential, to a cylinder at another potential

Here, the change in radial velocity is given by

$$\Delta v_r = v_{rf} = \int dz \frac{q E(r,z)}{m_0 v^2}$$

An einzel lens has three elements, no net acceleration, but still produces a focusing effect

The focusing effect is a complicated function of cylinder lengths, voltages, and diameters. Some plots of focal length can be found in Humphries, chapter 6, or in Harting and Read, Electrostatic Lenses (1976).
Solenoidal Magnetic Lens

A finite length solenoid also has a focusing effect.

The equations of motion of a particle moving along z, assuming gamma is constant since B can't change a particles energy:

\[ \gamma m_0 v_r = -q v_\theta B_z + \frac{q m_0 v_\theta^2}{r} \]

\[ \gamma m_0 v_\theta = -q v_z B_r - \frac{q m_0 v_r v_\theta}{r} \]

Take \( r \sim \) constant, and a small net rotation, so \( v_\theta \) small. Can then drop the Coriolis force.

\[ \frac{v_\theta}{\gamma} \frac{dv_\theta}{dz} = + \frac{e v_\theta}{\gamma m_0} \left( \frac{r}{z} \right) \frac{dB_z(0,z)}{dz} \]

\[ v_\theta = -\frac{q r B_z}{2 \gamma m_0} = \text{const.} = 0 \]
Quadrupole Lenses

Electrostatic quadrupole (from Humphries)

\[
\frac{1}{2} \frac{dV_r}{dr} = -\frac{q}{\varepsilon_0 m_0} \frac{B_z^2}{2 \varepsilon_0 m_0} \frac{r}{m_0} \left( \frac{q}{2 \varepsilon_0 m_0} \right)^2
\]

\[
\frac{dV_r}{dr} = -\frac{r}{2V_z} \left( \frac{q B_z^2}{\varepsilon_0 m_0} \right) \quad \text{integrate}
\]

\[
\frac{V_r}{V_z} = \frac{r}{r'} = -\frac{r}{4} \int d\zeta \left( \frac{q B_z(0, \zeta)^2}{\varepsilon_0 m_0 V_z} \right)
\]

\[
f = \frac{r}{r'} = 4 \left[ \int d\zeta \left( \frac{q B_z(0, \zeta)^2}{\varepsilon_0 m_0 V_z} \right) \right]^{-1}
\]

equipotential, red negative, blue positive, y - up, x - to right

electric field, red negative, blue positive, y - up, x - to right
In both cases, want field like the following (a is distance from center to nearest electrode or pole surface)

\[
E_x = \frac{E_0 x}{a}
\]

\[
E_y = -\frac{E_0 y}{a}
\]

\[
\frac{\phi(x, y)}{E_0 a^2} = \left(\frac{y}{a}\right)^2 - \left(\frac{x}{a}\right)^2
\]

For the magnetic quadrupole, the particle orbit equations are given by

\[
\frac{d^2 y}{d\tau^2} = \left(\frac{\gamma \beta_0}{\gamma m_0 a v_x}\right) y
\]

\[
\frac{d^2 x}{d\tau^2} = -\left(\frac{\gamma \beta_0}{\gamma m_0 a v_x}\right) x
\]
particle orbit solutions, when passing axially through the quadrupole are given below, with $x_1, y_1, x'_1, y'_1$ being the initial positions and angles

\[
x(z) = x_i \cos \sqrt{K_m} z + \frac{x_1'}{\sqrt{K_m}} \sin \sqrt{K_m} z
\]

\[
x'(z) = -x_i \sqrt{K_m} \sin \sqrt{K_m} z + x_1' \cos \sqrt{K_m} z
\]

\[
y(z) = y_i \cosh \sqrt{K_m} z + \frac{y_1'}{\sqrt{K_m}} \sinh \sqrt{K_m} z
\]

\[
y'(z) = y_i \sqrt{K_m} \sinh \sqrt{K_m} z + y_1' \cosh \sqrt{K_m} z
\]

these orbits are equivalent for magnetic or electric quadrupoles. Only the constant kappa changes for one case or the other.

\[
K_m = \frac{q B_0}{\gamma m_0 a v_z^2}
\]

\[
K_e = \frac{q E_0}{\gamma m_0 a v_z^2}
\]

Lens parameters, with $x$ -> focusing and $y$ -> defocusing, are shown below, if the quadrupole is rotated 90 degrees, $x$ is defocusing and $y$ is focusing.

\[
f_i = \left( \frac{\sqrt{m_0}}{m_0} \sin \sqrt{K_m} l \right)^{-1}
\]

\[
h_i = (1 - \cos \sqrt{K_m} l) f_i
\]

\[
f_i = -\left( \frac{\sqrt{m_0}}{m_0} \sinh \sqrt{K_m} l \right)^{-1}
\]

\[
h_i = (1 - \cosh \sqrt{K_m} l) f_i
\]